

## Kinematic Models of Mobile Robots

### Assumptions:

- The robot moves in a planar surface.
- The guidance axis are perpendicular to floor.
- Wheels rotate without any slippery problems.
- The robot does not have flexible parts.
- During small amounts of time, which direction is maintained constant, the vehicle will move from one point to other following a circumference arc.
- The robot is considered as a solid rigid body, and any movable parts are the direction wheels, which are moved following a commanded control position.

### Kinematic restrictions.

Consider an inverted pendulum as shown in figure 1. Its movement is restricted by the following equation

$$x^2 + y^2 - l^2 = 0 \quad (1)$$

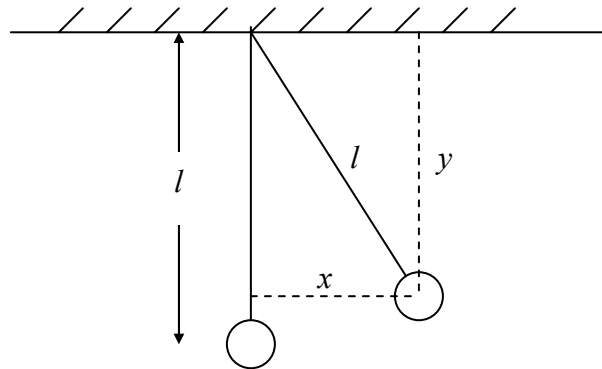


Figure 1. Simple pendulum

Similar restrictive equations are found in kinematic equations of mobile robots.

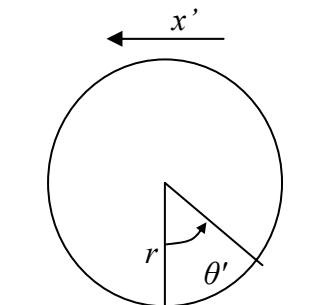


Figure 2. Speed of wheels.

Wheel movement (speed) in direction  $x$  is calculated by radius  $r$  and angle speed rotation  $\theta$  by.

$$x' = r \theta' \quad (2)$$

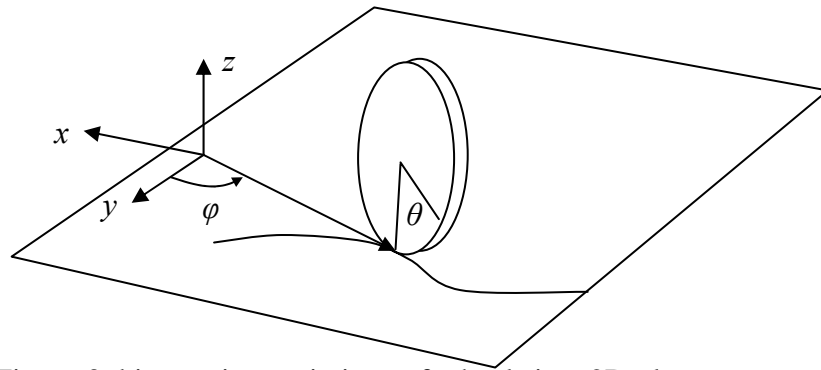


Figure 3. kinematic restrictions of wheels in a 2D plane.

This is, the speed in the  $x$  direction is directly proportional by the angular velocity of the wheel. However, other restrictions appear in wheels when the movement is restricted to a 2D plane  $(x,y)$ . Assume the angular orientation of a wheel is defined by angle  $\varphi$ . Then, while the wheel is following a path and having no slippery conditions, the velocity of the wheel at a given time, which is set by  $r\theta'$ , has the following restrictive velocity components  $(x',y')$  with respect to coordinate axes  $X$  and  $Y$ .

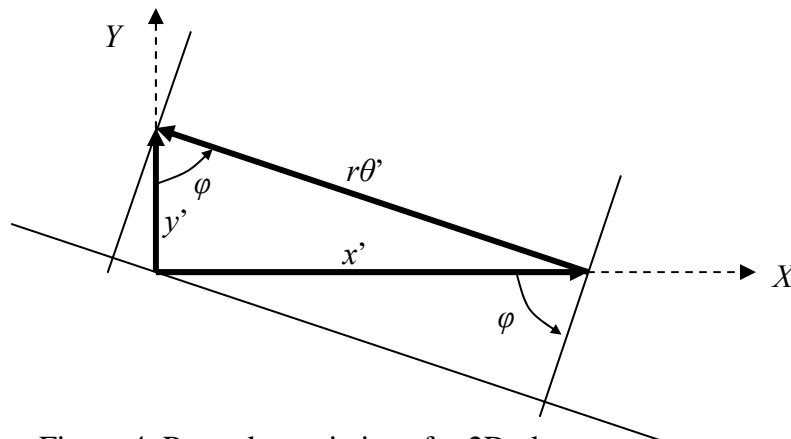


Figure 4. Remark restrictions for 2D plane movement.

$$r\theta' = -x' \sin \varphi + y' \cos \varphi \quad (3)$$

$$0 = x' \cos \varphi + y' \sin \varphi \quad (4)$$

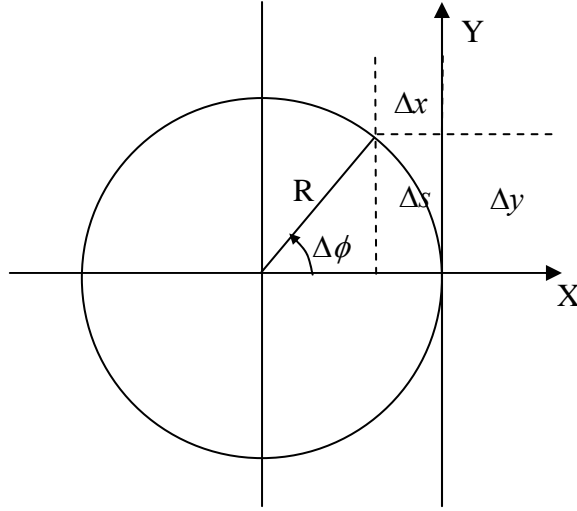


Figure 5. Circumference movement of the vehicle.

Consider now that the mobile robot (or vehicle) follows a circular trajectory as shown in figure 5. Notice that the lineal and angular velocities of the vehicle are given by

$$v = \frac{\Delta s}{\Delta t} \quad (5)$$

and

$$\omega = \frac{\Delta \phi}{\Delta t} \quad (6)$$

where  $\Delta s$  and  $\Delta \phi$  are the arc distance traveled by the wheel, and its respective orientation with respect to the global coordinates.

The arc distance  $\Delta s$  traveled in  $\Delta t$  time is obtained by:

$$\Delta s = R \Delta \phi \quad (7)$$

where  $R$  is the circumference radius of the wheel.

The *curvature* is defined as the inverse of the radius  $R$  as:

$$\gamma = \frac{1}{R} = \frac{\Delta \phi}{\Delta s} \quad (8)$$

The movement equations in the initial position are given by the following two expressions:

$$(\Delta x) = R(\cos(\Delta \phi) - 1) \quad (9)$$

$$(\Delta y) = R \sin(\Delta \phi) \quad (10)$$

An extension of the later equations is provided in the next expressions, considering a specific initial orientation of angle  $\phi$ . This is accomplished by rotating the earlier initial coordinates (9) and (10).

$$\Delta x = R[\cos(\Delta \phi) - 1] \cos \phi - R \sin(\Delta \phi) \sin \phi \quad (11)$$

$$\Delta y = R[\cos(\Delta\phi) - 1]\sin\phi + R\sin(\Delta\phi)\cos\phi \quad (12)$$

Assuming now that the control interval is sufficiently small, then we can assume that the orientation change would be small enough, and

$$\cos(\Delta\phi) \cong 1 \quad (13)$$

$$\sin(\Delta\phi) \cong \Delta\phi \quad (14)$$

Substituting (13) and (14) into (11) and (12), we got

$$\Delta x = -R\Delta\phi\sin\phi \quad (15)$$

$$\Delta y = R\Delta\phi\cos\phi \quad (16)$$

Now, considering (7), we got

$$\Delta x = -\Delta s\sin\phi \quad (17)$$

$$\Delta y = \Delta s\cos\phi \quad (18)$$

Dividing both sides of equations (17) and (18) by  $\Delta t$ , and considering also (5), if  $\Delta t$  tends to zero, we finally got

$$x' = -v\sin\phi \quad (17)$$

$$y' = v\cos\phi \quad (18)$$

Also, using (6) we can obtain the complimentary equation

$$\phi' = \varpi \quad (19)$$

### Jacobian Model.

Assume that  $p$  represents a point in the space having  $n$  generalized coordinates, and  $q$  a vector of  $m$  actuation variables (for  $n > m$ ), and assume  $p'$  and  $q'$  are the respective derivatives of such vectors, then the direct model is obtained by the Jacobian matrix,  $J(p)$  by

$$p' = J(p)q' \quad (20)$$

This jacobian expression can be written in the form (Zhao and Bennet):

$$p' = f(p) + \sum_{i=1}^m g(p)_i q'_i \quad (21)$$

$$p' = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varpi \quad (22)$$

where  $v$  is the linear velocity of the vehicle, and  $\varpi$  is its angular velocity. These equations can also be changed to the form of equation (20) as:

$$\begin{bmatrix} x' \\ y' \\ \phi' \end{bmatrix} = \begin{bmatrix} -\sin\phi & 0 \\ \cos\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \varpi \end{bmatrix} \quad (23)$$

for  $q' = [v \ \varpi]^T$

Notice that combining the first two equations from (23), and eliminating  $v$ , we got back the restricted relationship (4) by

$$x' \cos \phi + y' \sin \phi = 0 \quad (24)$$

This is due that the vehicle can only move along its longitudinal axis by

$$\text{tg } \phi = -\frac{x'}{y'} \quad (25)$$

In other words, the vehicle position  $(x,y)$  and its orientation  $\phi$  are not independent.

The inverse model of the system involves the inverse of the Jacobian. When the Jacobian is not a square matrix, it is necessary to calculate its pseudoinverse, by multiplying both sides by  $J^T$ , and solving for  $q'$ , to obtain:

$$q' = \{J(p)^T J(p)\}^{-1} J(p)^T p' \quad (26)$$

Then, for model (23) and using (20), we obtain

$$\begin{bmatrix} v \\ \varpi \end{bmatrix} = \begin{bmatrix} -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ \phi' \end{bmatrix} \quad (27)$$

From the first relationship, we obtain the earlier restricted condition (3) by

$$v = -x' \sin \phi + y' \cos \phi \quad (28)$$

### Configurations of Mobile Robots.

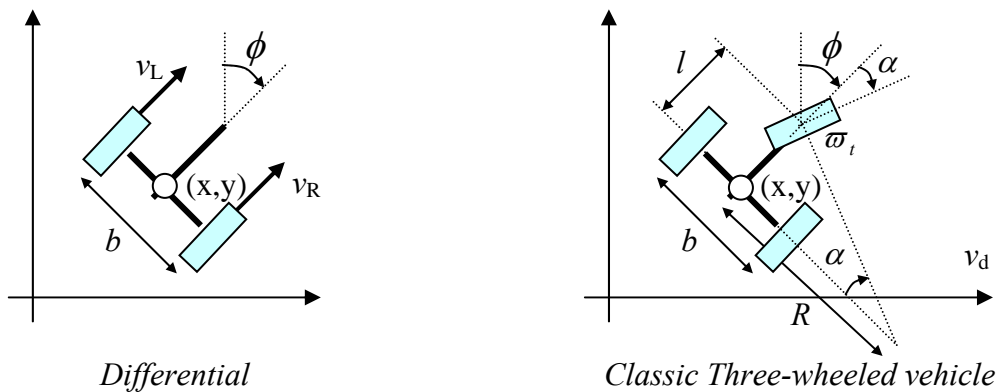


Figure 6. Typical mobile robot configurations.

The two sketches shown in figure 6 show the differential and the classical three-wheeled vehicles. The differential configuration use independent velocities in both wheels left and right ( $v_L$ , and  $v_R$ , respectively) to move in the 2D plane to a specific point  $(x,y)$  and specific orientation  $\phi$ . The three wheeled vehicle uses a single controlled angle and speed wheel to move to a desired position and orientation.

### Differential Configuration.

Assume for differential configuration model, that  $\varpi_L$  and  $\varpi_R$  are the corresponding angular velocities of the left and right wheels. Given the radius of the wheels as  $r$ , the corresponding linear and angular velocities of the vehicle are given by

$$v = \frac{v_R + v_L}{2} = \frac{\varpi_R + \varpi_L}{2} r \quad (29)$$

$$\varpi = \frac{v_R - v_L}{b} = \frac{\varpi_R - \varpi_L}{b} r \quad (30)$$

where  $b$  is the *bias* of the vehicle (separation of the two central wheels). Also, if the linear and angular velocities are provided, then the angular velocities of the wheels can be obtained by

$$\varpi_L = \frac{v - (b/2)\varpi}{r} \quad (31)$$

$$\varpi_R = \frac{v + (b/2)\varpi}{r} \quad (32)$$

Substituting equations (29) and (30) into the model of mobile robots (22), we found

$$\begin{bmatrix} x' \\ y' \\ \phi' \end{bmatrix} = \begin{bmatrix} -(r \sin \phi)/2 \\ (r \cos \phi)/2 \\ -r/b \end{bmatrix} \varpi_L + \begin{bmatrix} -(r \sin \phi)/2 \\ (r \cos \phi)/2 \\ r/b \end{bmatrix} \varpi_R \quad (33)$$

$$\begin{bmatrix} x' \\ y' \\ \phi' \end{bmatrix} = \begin{bmatrix} -(r \sin \phi)/2 & -(r \sin \phi)/2 \\ (r \cos \phi)/2 & (r \cos \phi)/2 \\ -r/b & r/b \end{bmatrix} \begin{bmatrix} \varpi_L \\ \varpi_R \end{bmatrix} \quad (34)$$

### Three-wheeled Configuration.

This configuration is the Romeo 3R configuration. For this vehicle the control angle for direction is defined by angle  $\alpha$  (or by its angular velocity  $\varpi_\alpha$ ), and the angular velocity of the wheel itself  $\varpi_t$  (or by its total velocity  $v_t = r\varpi_t$ ). Assume that the guidance point of the vehicle is in the back part of the control wheel (central back axis). For this configuration, the corresponding model is obtained by

$$v = v_t \cos \alpha = r \varpi_t \cos \alpha \quad (35)$$

and

$$\alpha' = \varpi_\alpha \quad (36)$$

Also, the angular velocity orientation is given by

$$\phi' = \frac{r \varpi_t}{l} \sin \alpha = \frac{v_t}{l} \sin \alpha \quad (37)$$

Substituting these equations into the model (22), we found the model by

$$\begin{bmatrix} x' \\ y' \\ \phi' \\ \alpha' \end{bmatrix} = \begin{bmatrix} -\sin \phi \cos \alpha \\ \cos \phi \cos \alpha \\ (\sin \alpha)/l \\ 0 \end{bmatrix} v_t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \varpi_\alpha = \begin{bmatrix} -\sin \phi \cos \alpha & 0 \\ \cos \phi \cos \alpha & 0 \\ (\sin \alpha)/l & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_t \\ \varpi_\alpha \end{bmatrix} \quad (38)$$

Notice that once known the desired lineal  $v$  and angular velocities  $\varpi$  of the vehicle (as shown in (27), the control variables  $\alpha$  and  $\varpi_t$  can be obtained by

$$\alpha = \arctan\left(\frac{l}{R}\right) = \arctan\left(\frac{l\varpi}{v}\right) \quad (39)$$

$$\varpi_t = \frac{v_t}{r} = \frac{\sqrt{v^2 + \varpi^2 l^2}}{r} \quad (40)$$