

1. Poisson's equation

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In [mathematics](#), **Poisson's equation** is a [partial differential equation](#) with broad utility in [electrostatics](#), [mechanical engineering](#) and [theoretical physics](#). It is named after the [French mathematician, geometer and physicist Siméon-Denis Poisson](#). The Poisson equation is

$$\Delta\varphi = f$$

where Δ is the [Laplace operator](#), and f and φ are [real](#) or [complex](#)-valued [functions](#) on a [manifold](#). When the manifold is [Euclidean space](#), the Laplace operator is often denoted as ∇^2 and so Poisson's equation is frequently written as

$$\nabla^2\varphi = f.$$

In three-dimensional [Cartesian coordinates](#), it takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi(x, y, z) = f(x, y, z).$$

For vanishing f , this equation becomes [Laplace's equation](#)

$$\Delta\varphi = 0.$$

The Poisson equation may be solved using a [Green's function](#); a general exposition of the Green's function for the Poisson equation is given in the article on the [screened Poisson equation](#). There are various methods for numerical solution. The [relaxation method](#), an iterative algorithm, is one example.

[[edit](#)] Electrostatics

Main article: [Electrostatics](#)

One of the cornerstones of [electrostatics](#) is the posing and solving of problems that are described by the Poisson equation. Finding φ for some given f is an important practical problem, since this is the usual way to find the [electric potential](#) for a given [charge](#) distribution.

The derivation of Poisson's equation in electrostatics follows. [SI](#) units are used and Euclidean space is assumed.

Starting with [Gauss' law](#) for electricity (also part of [Maxwell's equations](#)) in a differential control volume, we have:

$$\nabla \cdot \mathbf{D} = \rho_f$$

∇ is the [divergence operator](#).

\mathbf{D} is the [electric displacement field](#).

ρ_f is the [free charge density](#) (describing charges brought from outside).

Assuming the medium is linear, isotropic, and homogeneous (see [polarization density](#)), then:

$$\mathbf{D} = \epsilon \mathbf{E}$$

ϵ is the [permittivity](#) of the medium.

\mathbf{E} is the [electric field](#).

By substitution and division, we have:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}$$

In the absence of a changing magnetic field, \mathbf{B} , [Faraday's law of induction](#) gives:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0$$

$\nabla \times$ is the [curl operator](#).

t is time.

Since the [curl](#) of the electric field is zero, it is defined by a scalar electric potential field, φ (see [Helmholtz decomposition](#)).

$$\mathbf{E} = -\nabla \varphi$$

Eliminating \mathbf{E} by substitution, we have a form of the Poisson equation:

$$\nabla \cdot \nabla \varphi = \nabla^2 \varphi = -\frac{\rho}{\epsilon}.$$

Solving Poisson's equation for the potential requires knowing the charge density distribution. If the charge density is zero, then [Laplace's equation](#) results. If the charge density follows a [Boltzmann distribution](#), then the [Poisson-Boltzmann equation](#) results. The Poisson-Boltzmann equation plays a role in the development of the [Debye-Hückel theory of dilute electrolyte solutions](#).

(Note: Although the above discussion assumes that the magnetic field is not varying in time, the same Poisson equation arises even if it does vary in time, as long as the [Coulomb gauge](#) is used. However, in this more general context, computing φ is no longer sufficient to calculate \mathbf{E} , since the latter also depends on the [magnetic vector potential](#), which must be independently computed.)

[\[edit\]](#) Potential of a Gaussian charge density

If there is a static spherically symmetric [Gaussian](#) charge density $\rho(r)$:

$$\rho(r) = \frac{Q}{\sigma^3 \sqrt{2\pi}^3} e^{-r^2/(2\sigma^2)},$$

where Q is the total charge, then the solution $\varphi(r)$ of Poisson's equation,

$$\nabla^2 \varphi = -\frac{\rho}{\epsilon},$$

is given by ^[citation needed]

$$\varphi(r) = \frac{1}{4\pi\epsilon} \frac{Q}{r} \operatorname{erf}\left(\frac{r}{\sqrt{2}\sigma}\right)$$

where $\operatorname{erf}(x)$ is the [error function](#). This solution can be checked explicitly by a careful manual evaluation of $\nabla^2 \varphi$. Note that, for r much greater than σ , the erf function approaches unity and the potential $\varphi(r)$

approaches the [point charge](#) potential $\frac{1}{4\pi\epsilon_0} \frac{Q}{r}$, as one would expect. Furthermore the erf function approaches 1 extremely quickly as its argument increases; in practice for $r > 3\sigma$ the relative error is smaller than one part in a thousand.

2. Relaxation method

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This article is about a mathematical concept. For other uses, see [Relaxation](#).

In [numerical mathematics](#), the **relaxation method** is a method for obtaining numerical approximations to the solutions of systems of equations, including certain types of [elliptic partial differential equations](#), in particular [Laplace's equation](#) and its generalization, [Poisson's equation](#). The function is assumed to be given on the boundary of a shape, and has to be computed on its interior.

This relaxation method should not be confused with the unrelated [relaxation technique](#) in mathematical optimization.

[\[edit\]](#) Sketch

When φ is a smooth real-valued function on the real numbers, its second derivative can be approximated by:

$$\frac{d^2}{dx^2} \varphi(x) = h^{-2} (\varphi(x-h) - 2\varphi(x) + \varphi(x+h)) + \mathcal{O}(h^2).$$

Using this in both dimensions for a function φ of two arguments at the point (x, y) , and solving for $\varphi(x, y)$, results in:

$$\varphi(x, y) = \frac{1}{4} (\varphi(x+h, y) + \varphi(x, y+h) + \varphi(x-h, y) + \varphi(x, y-h) - h^2 \nabla^2 \varphi(x, y)) -$$

To approximate the solution of the Poisson equation:

$$\nabla^2 \varphi = f$$

numerically on a two-dimensional grid with grid spacing h , the relaxation method assigns the given values of function φ to the grid points near the boundary and arbitrary values to the interior grid points, and then repeatedly performs the assignment $\varphi := \varphi^*$ on the interior points, where φ^* is defined by:

$$\varphi^*(x, y) = \frac{1}{4} (\varphi(x+h, y) + \varphi(x, y+h) + \varphi(x-h, y) + \varphi(x, y-h) - h^2 f(x, y)) ,$$

until convergence.

The method, sketched here for two dimensions, is readily generalized to other numbers of dimensions.

[edit] Convergence and acceleration

While the method always converges, it does so very slowly. To speed up the computation, one can first compute an approximation on a coarser grid – usually the double spacing $2h$ – and use that solution with [interpolated](#) values for the other grid points as the initial assignment. This can then also be done recursively for the coarser computation.

[edit] See also

- The [Jacobi method](#) is a simple relaxation method.
- The [Gauss–Seidel method](#) is an improvement upon the Jacobi method.
- [Successive over-relaxation](#) can be applied to either of the Jacobi and Gauss–Seidel methods to speed convergence.

This [mathematical analysis](#)-related article is a [stub](#). You can help Wikipedia by [expanding it](#).

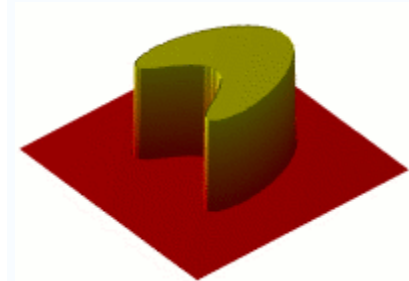


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3. Heat equation

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The heat equation predicts that if a hot body is placed in a box of cold water, the temperature of the body will decrease, and eventually (after infinite time, and subject to no external heat sources) the temperature in the box will equalize.

The **heat equation** is an important [partial differential equation](#) which describes the distribution of [heat](#) (or variation in temperature) in a given region over time. For a [function](#) $u(x,y,z,t)$ of three spatial variables (x,y,z) and the time variable t , the heat equation is

$$\frac{\partial u}{\partial t} - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

or equivalently

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

where α is a constant. For the mathematical treatment it is sufficient to consider the case $\alpha=1$.

The heat equation is of fundamental importance in diverse scientific fields. In [mathematics](#), it is the prototypical [parabolic partial differential equation](#). In [probability theory](#), the heat equation is connected with the study of [Brownian motion](#) via the [Fokker–Planck equation](#). The [diffusion equation](#), a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

Internal heat generation

The function u above represents temperature of a body. Alternatively, it is sometimes convenient to change units and represent u as the [heat density](#) of a medium. Since heat density is proportional to temperature in a homogeneous medium, the heat equation is still obeyed in the new units.

Suppose that a body obeys the heat equation and, in addition, generates its own heat per unit volume (e.g., in watts/L) at a rate given by a known function q varying in space and time.^[3] Then the heat per unit volume u satisfies an equation

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + q.$$

For example, a tungsten light bulb filament generates heat, so it would have a positive nonzero value for q when turned on. While the light is turned off, the value of q for the tungsten filament would be zero.

4. Diffusion equation

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The **diffusion equation** is a [partial differential equation](#) which describes density fluctuations in a material undergoing [diffusion](#). It is also used to describe processes exhibiting diffusive-like behaviour, for instance the 'diffusion' of [alleles](#) in a population in [population genetics](#).

The equation is usually written as:

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} = \nabla \cdot [D(\phi, \mathbf{r}) \nabla \phi(\mathbf{r}, t)],$$

where $\phi(\mathbf{r}, t)$ is the [density](#) of the diffusing material at location \mathbf{r} and time t and $D(\phi, \mathbf{r})$ is the collective [diffusion coefficient](#) for density ϕ at location \mathbf{r} ; the [nabla symbol](#) ∇ represents the vector [differential operator del](#) acting on the space coordinates. If the diffusion coefficient depends on the density then the equation is nonlinear, otherwise it is linear. If D is constant, then the equation reduces to the following linear equation:

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} = D \nabla^2 \phi(\mathbf{r}, t),$$

also called the [heat equation](#). More generally, when D is a symmetric positive definite matrix, the equation describes [anisotropic](#) diffusion, which is written (for three dimensional diffusion) as:

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial}{\partial x_i} \left[D_{ij}(\phi, \mathbf{r}) \frac{\partial \phi(\mathbf{r}, t)}{\partial x_j} \right]$$

[\[edit\]](#) Discretization

See also: [Discrete Gaussian kernel](#)

The diffusion equation is continuous in both time and space. One may discretize space, time, or both space and time, which arise in application. Discretizing time alone just corresponds to taking time slices of the continuous system, and no new phenomena arise. In discretizing space alone, the [Green's function](#) becomes the [discrete Gaussian kernel](#), rather than the continuous [Gaussian kernel](#). In discretizing both time and space, one obtains the [random walk](#).

[\[edit\]](#) Discretization (Image)

The product rule is used to rewrite the anisotropic tensor diffusion equation, in standard discretization schemes. Because direct discretization of the diffusion equation with only first order spatial central differences leads to checkerboard artifacts. The rewritten diffusion equation used in image filtering :

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} = \operatorname{div} \left[D(\phi, \mathbf{r}) \nabla \phi(\mathbf{r}, t) + \operatorname{trace} \left[D(\phi, \mathbf{r}) (\nabla \nabla^T \phi(\mathbf{r}, t)) \right] \right]$$

In which in image filtering $D(\phi, \mathbf{r})$ are symmetric matrices constructed from the [eigenvectors](#) of the image [structure tensors](#) . The spatial derivatives can then be approximated by two first order and a second order central [finite differences](#). The resulting diffusion algorithm can be written as an image [convolution](#) with an varying kernel (stencil) of size 3 x 3 in 2D and 3x3x3 in 3D.

5. Fokker–Planck equation

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A solution to the one-dimensional Fokker–Planck equation, with both the drift and the diffusion term. The initial condition is a [Dirac delta function](#) in $x = 1$, and the distribution drifts towards the left.

The **Fokker–Planck equation** describes the [time evolution](#) of the [probability density function](#) of the position of a particle, and can be generalized to other observables as well.^[1] It is named after [Adriaan Fokker](#) and [Max Planck](#) and is also known as the **Kolmogorov forward equation**, named after [Andrey Kolmogorov](#). The first use of the *Fokker–Planck* equation was for the statistical description of [Brownian motion](#) of a particle in a [fluid](#). The first consistent microscopic derivation of the Fokker–Planck equation in the single scheme of classical and quantum mechanics was performed^[2] by [Nikolay Bogoliubov](#) and [Nikolay Krylov](#).^[3]

In one spatial dimension x , the Fokker–Planck equation for a process with drift $D_1(x,t)$ and diffusion $D_2(x,t)$ is

$$\frac{\partial}{\partial t} f(x, t) = -\frac{\partial}{\partial x} [D_1(x, t) f(x, t)] + \frac{\partial^2}{\partial x^2} [D_2(x, t) f(x, t)].$$

More generally, the time-dependent probability distribution may depend on a set of N macrovariables x_i . The general form of the Fokker–Planck equation is then

$$\frac{\partial f}{\partial t} = - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left[D_i^1(x_1, \dots, x_N) f \right] + \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} \left[D_{ij}^2(x_1, \dots, x_N) f \right],$$

where D^1 is the drift [vector](#) and D^2 the [diffusion tensor](#); the latter results from the presence of the stochastic force.

[\[edit\]](#) Schrödinger equation for a free particle

Main article: [Schrödinger equation](#)

With a simple division, the [Schrödinger equation](#) for a single particle of [mass](#) m in the absence of any applied force field can be rewritten in the following way:

$$\psi_t = \frac{i\hbar}{2m} \Delta \psi, \text{ where } i \text{ is the } \text{unit imaginary number}, \text{ and } \hbar \text{ is } \text{Planck's constant} \text{ divided by } 2\pi, \text{ and } \psi \text{ is the } \text{wavefunction} \text{ of the particle.}$$

This equation is formally similar to the particle diffusion equation, which one obtains through the following transformation:

$$\begin{aligned} c(\vec{R}, t) &\rightarrow \psi(\vec{R}, t) \\ D &\rightarrow \frac{i\hbar}{2m}. \end{aligned}$$

Applying this transformation to the expressions of the Green functions determined in the case of particle diffusion yields the Green functions of the [Schrödinger equation](#), which in turn can be used to obtain the [wavefunction](#) at any time through an integral on the [wavefunction](#) at $t=0$:

$$\begin{aligned} \psi(\vec{R}, t) &= \int \psi(\vec{R}^0, t=0) G(\vec{R} - \vec{R}^0, t) dR_x^0 dR_y^0 dR_z^0, \text{ with} \\ G(\vec{R}, t) &= \left(\frac{m}{2\pi i \hbar t} \right)^{3/2} e^{-\frac{\vec{R}^2 m}{2i\hbar t}}. \end{aligned}$$

Remark: this analogy between quantum mechanics and diffusion is a purely formal one. Physically, the evolution of the [wavefunction](#) satisfying [Schrödinger's equation](#) might have an origin other than diffusion.

6. Finite difference

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A **finite difference** is a mathematical expression of the form $f(x + b) - f(x + a)$. If a finite difference is divided by $b - a$, one gets a [difference quotient](#). The approximation of derivatives by finite differences plays a central role in [finite difference methods](#) for the [numerical](#) solution of [differential equations](#), especially [boundary value problems](#).

[Recurrence relations](#) can be written as difference equations by replacing iteration notation with finite differences.

[\[edit\]](#) Forward, backward, and central differences

Only three forms are commonly considered: forward, backward, and central differences.

A **forward difference** is an expression of the form

$$\Delta_h[f](x) = f(x + h) - f(x).$$

Depending on the application, the spacing h may be variable or held constant.

A **backward difference** uses the function values at x and $x - h$, instead of the values at $x + h$ and x :

$$\nabla_h[f](x) = f(x) - f(x - h).$$

Finally, the **central difference** is given by

$$\delta_h[f](x) = f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h).$$

[\[edit\]](#) Relation with derivatives

The [derivative](#) of a function f at a point x is defined by the [limit](#)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

If h has a fixed (non-zero) value, instead of approaching zero, then the right-hand side is

$$\frac{f(x + h) - f(x)}{h} = \frac{\Delta_h[f](x)}{h}.$$

Hence, the forward difference divided by h approximates the derivative when h is small. The error in this approximation can be derived from [Taylor's theorem](#). Assuming that f is continuously differentiable, the error is

$$\frac{\Delta_h[f](x)}{h} - f'(x) = O(h) \quad (h \rightarrow 0).$$

The same formula holds for the backward difference:

$$\frac{\nabla_h[f](x)}{h} - f'(x) = O(h).$$

However, the central difference yields a more accurate approximation. Its error is proportional to square of the spacing (if f is twice continuously differentiable):

$$\frac{\delta_h[f](x)}{h} - f'(x) = O(h^2).$$

The main problem with the central difference method, however, is that oscillating functions can yield zero derivative. If $h(nh)=1$ for n uneven, and $h(nh)=2$ for n even, then $f'(nh)=0$ if it is calculated with the central difference scheme. This is particularly troublesome if the domain of f is discrete.

[\[edit\]](#) Higher-order differences

In an analogous way one can obtain finite difference approximations to higher order derivatives and differential operators. For example, by using the above central difference formula for $f'(x + h/2)$ and $f'(x - h/2)$ and applying a central difference formula for the derivative of f' at x , we obtain the central difference approximation of the second derivative of f :

$$f''(x) \approx \frac{\delta_h^2[f](x)}{h^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

More generally, the n^{th} -order forward, backward, and central differences are respectively given by:

$$\begin{aligned} \Delta_h^n[f](x) &= \sum_{i=0}^n (-1)^i \binom{n}{i} f(x + (n-i)h), \\ \nabla_h^n[f](x) &= \sum_{i=0}^n (-1)^i \binom{n}{i} f(x - ih), \\ \delta_h^n[f](x) &= \sum_{i=0}^n (-1)^i \binom{n}{i} f\left(x + \left(\frac{n}{2} - i\right)h\right). \end{aligned}$$

Note that the central difference will, for odd n , have h multiplied by non-integers. This is often a problem because it amounts to changing the interval of discretization. The problem may be remedied taking the average of $\delta_h^n[f](x - h/2)$ and $\delta_h^n[f](x + h/2)$.

The relationship of these higher-order differences with the respective derivatives is very straightforward:

$$\frac{d^n f}{dx^n}(x) = \frac{\Delta_h^n[f](x)}{h^n} + O(h) = \frac{\nabla_h^n[f](x)}{h^n} + O(h) = \frac{\delta_h^n[f](x)}{h^n} + O(h^2).$$

Higher-order differences can also be used to construct better approximations. As mentioned above, the first-order difference approximates the first-order derivative up to a term of order h . However, the combination

$$\frac{\Delta_h[f](x) - \frac{1}{2}\Delta_h^2[f](x)}{h} = \frac{f(x+2h) - 4f(x+h) + 3f(x)}{2h}$$

approximates $f'(x)$ up to a term of order h^2 . This can be proven by expanding the above expression in [Taylor series](#), or by using the calculus of finite differences, explained below.

If necessary, the finite difference can be centered about any point by mixing forward, backward, and central differences.

[\[edit\]](#) **n -th difference**

The n th forward difference of a function $f(x)$ is given by

$$\Delta^n[f](x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(x+k)$$

where $\binom{n}{k}$ is the [binomial coefficient](#). Forward differences applied to a [sequence](#) are sometimes called the [binomial transform](#) of the sequence, and have a number of interesting combinatorial properties.

Forward differences may be evaluated using the [Nörlund–Rice integral](#). The integral representation for these types of series is interesting because the integral can often be evaluated using [asymptotic expansion](#) or [saddle-point](#) techniques; by contrast, the forward difference series can be extremely hard to evaluate numerically, because the binomial coefficients grow rapidly for large n .

[\[edit\]](#) **Finite difference in several variables**

Finite differences can be considered in more than one variable. They are analogous to [partial derivatives](#) in several variables.

Some partial derivative approximations are:

$$f_x(x, y) \approx \frac{f(x+h, y) - f(x-h, y)}{2h}$$

$$f_y(x, y) \approx \frac{f(x, y+k) - f(x, y-k)}{2k}$$

$$f_{xx}(x, y) \approx \frac{f(x+h, y) - 2f(x, y) + f(x-h, y)}{h^2}$$

$$f_{yy}(x, y) \approx \frac{f(x, y + k) - 2f(x, y) + f(x, y - k)}{k^2}$$

$$f_{xy}(x, y) \approx \frac{f(x + h, y + k) - f(x + h, y - k) - f(x - h, y + k) + f(x - h, y - k)}{4hk}$$