EE 4343/5329 - Control System Design Project
LECTURE 5

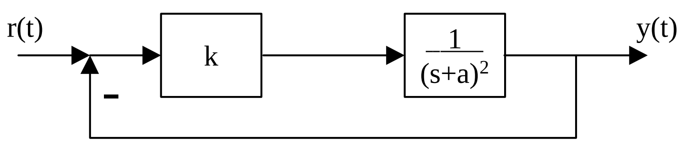
Updated: Friday, June 11, 1999

ROOT LOCUS DESIGN TECHNIQUE

Suppose the closed-loop transfer function depends on a design parameter *k*. We would like to know how the closed-loop poles vary with *k*.

The next example shows that the closed-loop poles vary continuously with *k* and trace out a path or *locus* in the *s*-plane. Thus, by varying *k* one actually moves the poles around in the *s*-plane.

Example 1- Variation of Closed-Loop Poles vs. a Design Gain

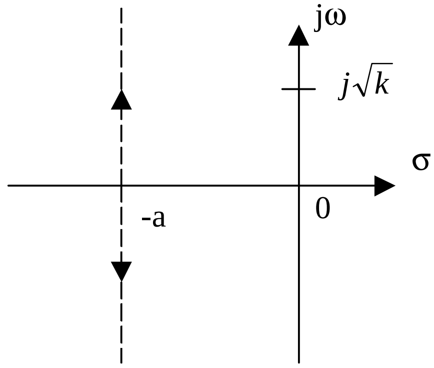


Let us find how the poles vary as the feedback gain *k* is increased from zero.

The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\dfrac{k}{(s+a)^2}}{1+\left(\dfrac{k}{(s+a)^2}\right)} = \frac{k}{(s+a)^2+k}.$$

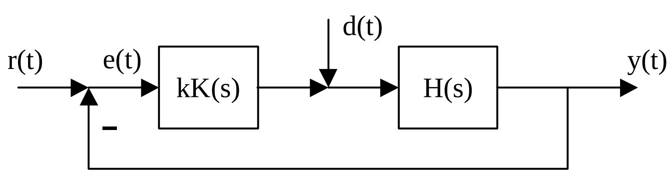
Compare this to the standard form $\Delta(s) = (s+a)^2 + b^2$ to see that, if $k>0$, the poles are at real part of $-a$ and imaginary part of $b = \sqrt{k}$. Thus, as *k* increases from zero, the poles simply move out along the line $s = -a$. This is illustrated in the figure



Root Locus (RL) Design Procedure

The root locus technique was developed by Evans working at North American Aviation during World War 2. The importance of this method lies in the fact that it allows one to see how the poles vary with *k without ever finding the actual roots*. Note that in the previous example, we actually computed the roots as a function of *k*. This is not necessary with root locus design. The root locus imparts a great deal of design insight about the structure of the control system.

A basic feedback control system is the TRACKING CONTROLLER given in the figure. The plant is H(s) and the compensator K(s); the feedback gain is k. The function of the tracker is to make the output y(t) follow the command or reference input r(t) by making the tracking error e(t)=r(t)-y(t) small. The disturbance d(t) plays no part here and is assumed equal to zero.



The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{kK(s)H(s)}{1 + kK(s)H(s)}\ .$$

In root locus design one focuses on the denominator

$$\Delta(s) = 1 + kK(s)H(s) \equiv 1 + kG(s)$$

where G(s) is the open-loop gain. Note that we use the same symbol for the denominator of T(s) as for the state-variable characteristic polynomial $\Delta(s) = |sI - A|$. However, 1+kG(s) is actually a polynomial fraction, whose numerator is the system characteristic polynomial.

Many design techniques rely on trying to **determine closed-loop properties from open-loop properties**. In root locus design, one uses the all-important open-loop gain *G(s)* to estimate the locations of the closed-loop poles, which are the roots of the numerator of $\Delta(s) = 1 + kG(s)$. The key point of RL design is that it is easier to plot the locations of the closed-loop poles versus the feedback gain parameter *k* than it is to find the actual closed-loop poles themselves. This was extremely important in days before digital computers when finding roots of high-order polynomials was difficult, and it also gives great insight into the properties of the closed-loop system.

In this discussion, one must distinguish between the closed-loop poles, which are the roots of the numerator of *D(s)= 1+kG(s)*, which depend on the feedback gain *k*, and the (open-loop) poles and zeros of G(s), which do not depend on k.

One also uses the RELATIVE DEGREE of *G(s)*. The relative degree of a polynomial fraction is the degree of the denominator minus the degree of the numerator. Since the total number of poles and zeros is the same, the number of zeros at infinity is equal to the relative degree. We consider the case when the polynomial fraction is PROPER, that is when the relative degree is greater than or equal to zero. The fraction is said to be STRICTLY PROPER if the relative degree is greater than or equal to 1.

Define G(s)= p(s)/q(s) and denote the relative degree of G(s) as m= deg(q)-deg(p). Then G(s) has m zeros at infinity.

The RL is a plot in the s-plane of the closed-loop poles as k varies from zero to infinity. To find the closed-loop poles we are interested in points where

$$1 + kG(s) = 0 ,$$

which can be written in several equivalent forms, including

$$G(s) = -\frac{1}{k}$$
$$q(s) + kp(s) = 0 .$$

All points in the s-plane that satisfy this equation are closed-loop poles for that value of k. Thus one notes that on the root locus, G(s) is real and negative.

In the textbook it is shown that several basic rules for plotting the root locus follow directly from these equations. These rules assume a positive gain k. The rules for negative k can likewise be derived.

<u>Basic Rules for Plotting the Root Locus</u>

1. As k goes from zero to infinity, the closed-loop poles move from the poles of G(s) to the zeros of G(s).

2. The loci are symmetric with respect to the real axis.

3. The real-axis to the left of an odd number of poles + zeros is on the RL.
   (Recall the discussion on the phase of a rational function on the real axis!)

4. As k becomes large, m loci approach infinity asymptotically. The asymptotes are at angles of

$$\pm\left(\frac{180^o + i360^o}{m}\right) \quad ; i = 0,1,2,...$$

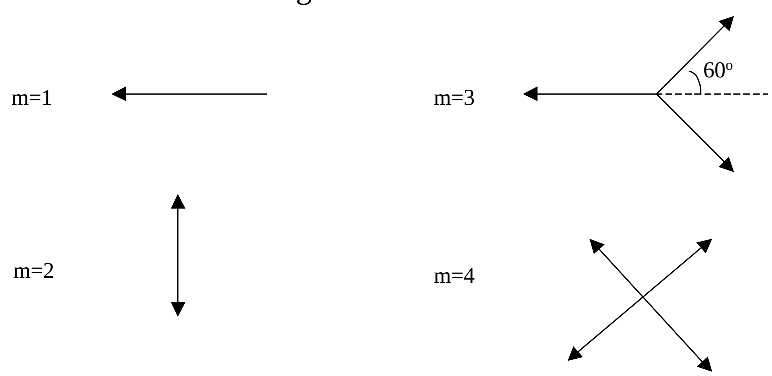
and they meet on the real axis at the centroid given by

$$c = \frac{\Sigma(\mathit{finite\ poles\ of\ G}) - \Sigma(\mathit{finite\ zeros\ of\ G})}{m}.$$

5. On the RL, G(s) is real and negative so that angle(G(s))= -180^o. This is useful for determining the 'angle of departure' of the RL from a specific open-loop pole of G.

There are other rules, including those for determining breakaway points from the real axis. However, MATLAB has a very good RL plotting routine which may be used to determine the RL in more detail than that given by these simple rules.

It is useful to note that the asymptotes in Rule 4 take on the following patterns for the first few values of the relative degree *m*.



Example 2- Redo Example 1 Using RL

In Example 1 we had

$$H(s) = \frac{1}{(s+a)^2}.$$

$$K(s) = 1$$

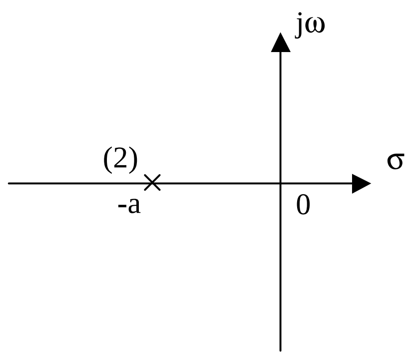
The closed-loop transfer function is

$$T(s)=\frac{Y(s)}{R(s)}=\frac{\dfrac{k}{(s+a)^2}}{1+\left(\dfrac{k}{(s+a)^2}\right)}=\frac{kK(s)H(s)}{1+kG(s)}$$

so that

$$G(s)=\frac{1}{(s+a)^2}\,.$$

The poles of *G(s)* are plotted in the figure. Note that there are two zeros at infinity.



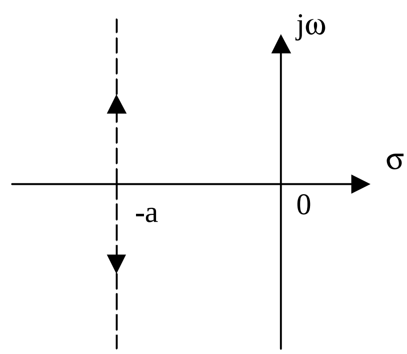
Now we apply the RL rules:

Rule 1. The closed-loop poles move from the x's to the zeros at infinity.

Rule 3. No part of the real axis is on the RL.

Rule 4. The relative degree is *m=2*, so the asymptotes are at $\pm 90^o$ as depicted in the figure above.

The RL is therefore determined to be as shown below.



Note that we did not find the closed-loop roots to draw the RL, in contrast to the procedure in Example 1. On the other hand, following the RL rules, we do not know that the imaginary part of the closed-loop poles is simply equal to $j\sqrt{k}$.

The RL shows its real power in more complex examples.

Example 3

Suppose the plant has transfer function

$$H(s) = \frac{s+5}{s^2}$$

and one uses the lag compensator

$$K(s) = \frac{s+8}{s+1}.$$

Then the loop gain is

$$G(s) = K(s)H(s) = \frac{(s+5)(s+8)}{s^2(s+1)}.$$

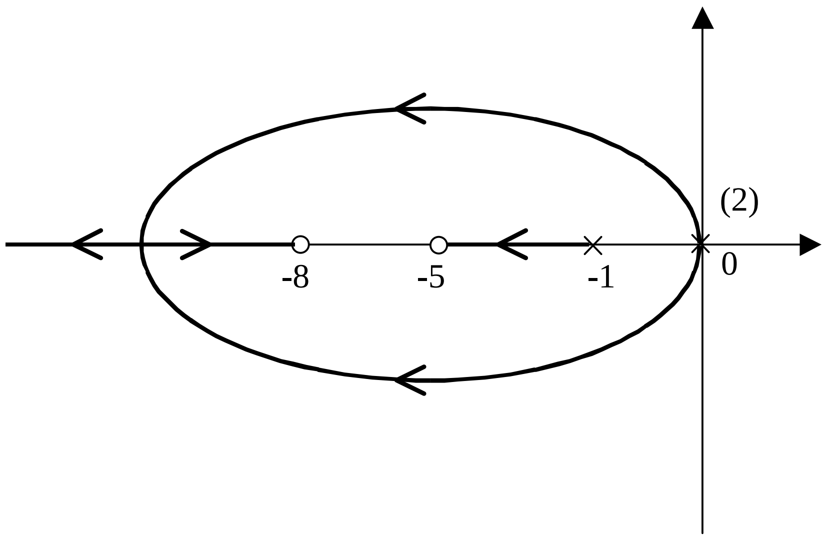
Now we apply the RL rules:

The RL moves from the x's to o's.

The real axis between *s=-1* and *s=-5* is on the RL. The real axis between s=-8 and $s = -\infty$ is on the RL.

The relative degree is *m=1*, so there is one zero at infinity. The asymptote is at -180^{o}, so the centroid need not be found.

The RL sketch found from these rules is given below.

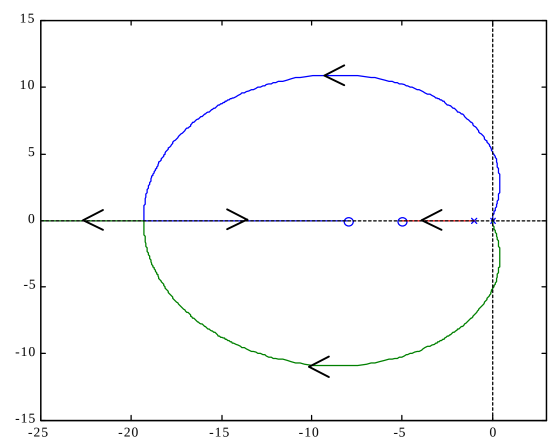


To plot the RL using MATLAB, one simply uses the command lines:

```
num= [1  13  40] ;
den= [1  1  0  0] ;
rlocus= (num,den)
```

The result is shown.

Note that the arrowheads were added using Powerpoint.



Note that the RL actually crosses the *jw* axis before coming to the left. This could be determined without MATLAB by using the *angle of departure rule*, which we will show in a later example.

The reason for this behavior is a loose rule based on experience that 'the root locus is repelled from poles'. Thus, the RL tries to 'get away from' the pole at *s=-1*. Once it does so, it 'sees' the zeros further to the left and heads that way.

The value of k for which the RL crosses the imaginary axis is easily found using the Routh test. The characteristic polynomial is found from 1+kG(s) as

$$s^2(s+1)+k(s+5)(s+8)=s^3+(k+1)s^2+13ks+40k\ .$$

The Routh Array is given by

$$\begin{array}{c|cc} s^3 & 1 & 13k \\ s^2 & k+1 & 40k \\ s^1 & 13k^2-27k & \\ s^0 & 40k & \end{array}$$

where one has multiplied the third row by k+1. To ensure that all coefficients in the first column are positive, one requires that $k > 27/13$. Thus, the RL crosses the imaginary axis at k= 27/13. The roots at this point are determined by substituting k= 27/13 into the characteristic polynomial and using the MATLAB root-finder 'roots'. The MATLAB command lines are:

```
k=27/13 ;
del=[1   k+1   13*k   40*k] ;
roots(del)
ans =
        0.0000 + 5.1962i
        0.0000 - 5.1962i
        -3.0769
```