EE 4343/5329 - Control System Design Project
LECTURE 10

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STATE-VARIABLE FEEDBACK (SVFB)

The linear state-space equations are given by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with $x(t) \in R^n$ the internal state, $u(t) \in R^m$ the control input, and $y(t) \in R^p$ the measured output. Matrix A is the system or plant matrix, B is the control input matrix, C is the output or measurement matrix, and D is the direct feed matrix. The open-loop poles are given by the roots of

$$\Delta(s) = |sI - A|$$

and the open-loop transfer function is given by

$$H(s) = C(sI - A)^{-1} B + D .$$

Feedback controllers may be designed using either the state description (A,B,C,D) or the input-output description H(s).

A simple feedback control scheme is to use the outputs to compute the control inputs according to the Proportional (P) feedback law

$$u = -Ky + v$$

where v(t) is the new external control input. With K the $m \times p$ proportional feedback gain matrix. Note that K has mp entries so that there are mp control loops. This sort of P feedback is sufficient to give the desired closed-loop stability and performance for many systems. This scheme is known as OUTPUT FEEDBACK (OPFB). OPFB design is not easy and we may discuss it later.

A more basic control scheme is to assume that ALL the states are measured as outputs, so that one may use the STATE-VARIABLE FEEDBACK (SVFB) control law

$$u = -Kx + v .$$

Feedback matrix K is $m \times n$ so that there are now mn control loops. Note that SVFB is the same as OPFB with C= I the identity matrix.

Assuming SVFB, the closed-loop system is given as

$$\dot{x} = (A - BK)x + Bv = A_c x + Bv$$

$$y = (C - DK)x + Dv = C_c + Dv$$

where A_c is the CLOSED-LOOP SYSTEM OR PLANT MATRIX and C_c is the closed-loop output matrix. The closed-loop poles are given by the roots of

$$\Delta_c(s) = |sI - A_c| = |sI - (A - BK)|$$

and the closed-loop transfer function is given by

$$H(s) = C_c(sI - A_c)^{-1}B + D = (C - DK)[sI - (A - BK)]^{-1}B + D.$$

Reachability

A system is said to be REACHABLE (sometimes called CONTROLLABLE) if the control input u(t) can be selected to drive the state x(t) from any given initial condition to any desired final value. This can be done if the input coupling in the system is sufficiently strong, which depends on the input-coupling matrix pair (A,B). If a system is not reachable, it can be made so by adding additional control inputs.

There is an easy test for reachability. Indeed, define the REACHABILITY MATRIX

$$U = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

Then, (A,B) is reachable iff U has full rank n. Note that U is an $n \times nm$ matrix so that it has more columns than rows if m>1. Such a matrix is called a flat matrix. (A matrix with more rows than columns is called a sharp matrix.)

If there is only one control input (the single-input (SI) case, where m=1), then U is square. In this case, it is easy to test whether U has rank n by making sure the determinant $|U|$ is nonzero. If m>1 one must find n linearly independent columns of U, which may be difficult particularly if the number of inputs m is large. In this case, define the REACHABILITY GRAMMIAN

$$G = UU^T$$

which is a square $n \times n$ matrix. Then the system is reachable iff $|G| \neq 0$.

Many design techniques rely on trying to **determine closed-loop properties from open-loop properties**. This is exactly the intent of the reachability test, which allows one to determine in terms of the open-loop matrices A and B what can be accomplished in the closed-loop system.

SVFB Pole Placement and Ackermann's Formula

Note that in the case of SVFB the output y(t) plays no role. This means that only matrices A and B will be important in SVFB. We would like to choose the feedback gain K so that the closed-loop characteristic polynomial $\Delta_c(s) = |sI - A_c| = |sI - (A - BK)|$ has prescribed roots. This is called the POLE-PLACEMENT problem. An important theorem says that the poles may be placed arbitrarily as desired iff (A,B) is reachable.

If the system is reachable, there are many techniques to find a suitable K that guarantees stability and/or places the poles. One technique that works for the SI case m=1 is ACKERMANN'S FORMULA

$$K = e_n U^{-1} \Delta_D(A),$$

where $e_n = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$ is the last row of the $n \times n$ identity matrix, and $\Delta_D(s)$ is the DESIRED characteristic polynomial. The desired characteristic polynomial evaluated at the plant matrix A is $\Delta_D(A)$, which is a MATRIX POLYNOMIAL.

Example 1

The angle subsystem of the inverted pendulum is given for some specific values of parameters by

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u$$

where the state is $x = [q \quad \dot{q}]$. The open-loop poles are given by the roots of

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & -1 \\ -9 & s \end{vmatrix} = s^2 - 9 = (s+3)(s-3),$$

which are s=-3, s=3. The system is unstable, with natural modes e^{-3t}, e^{3t}.

The design specifications are to design a SVFB K that will give a closed-loop POV of 4% with a settling time of τ_s= 1 sec. This will make the rod of the inverted pendulum balance upright. These specs. allow one to compute the desired closed-loop poles. In fact, since

$$t_s \approx 5t = 5/a$$

$$POV = 100 e^{-pz/\sqrt{1-z^2}}$$

one may find the required real-part and damping ratio of the closed loop poles to be $-a = -5$, $z \approx 1/\sqrt{2} = 0.707$. Thus, the natural frequency is $w_n = a/z$ = 7.072 so that the desired characteristic polynomial is

$\Delta_D = s^2 + 2as + w_n{}^2$ = $s^2+10s+50$.

To use Ackermann's formula, one first verifies reachability by computing the reachability matrix

$$U = [B \quad AB] = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

and checking that it is indeed nonsingular. Computing the quantities needed for Ackermann's formula now yields

$$U^{-1} = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}$$

$$\Delta_D(A) = A^2 + 10A + 50I = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 10 \\ 90 & 0 \end{bmatrix} + \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix} = \begin{bmatrix} 59 & 10 \\ 90 & 59 \end{bmatrix}.$$

Substituting now into Ackermann's formula yields the required SVFB of

$$K = e_n U^{-1} \Delta_D(A) = [0 \quad 1] \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix} \begin{bmatrix} 59 & 10 \\ 90 & 59 \end{bmatrix} = [-29.5 \quad -5].$$

This solves the problem.

Note that one may write the P SVFB as

$$u = -Kx + v = -[-29.5 \quad -5]x + v = [k_1 \quad k_2] \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + v = k_1 q + k_2 \dot{q} + v\,,$$

so that in effect a proportional-plus-derivative control is produced, with the proportional gain given by k_1 and the derivative gain by k_2.

To check the design, one should compute the actual closed-loop poles using

$$A_c = A - BK = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix} [-29.5 \quad -5] = \begin{bmatrix} 0 & 1 \\ -50 & -10 \end{bmatrix}$$

$$\Delta_c(s) = |sI - A_c| = \begin{vmatrix} s & -1 \\ 50 & s+10 \end{vmatrix} = s^2 + 10s + 50\,.$$

This is indeed equal to $\Delta_D(s)$.
