EE 4343/5329 - Control System Design Project
LECTURE 8

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ANALYSIS OF LINEAR STATE-SPACE SYSTEMS

We discuss the analysis and solution of linear time-invariant (LTI) state-variable systems. An example is provided at the end.

Frequency-Domain Solution

Many physical systems can be modeled in terms of the linear time-invariant (LTI) state-space equations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with $x(t) \in R^n$ the internal state, $u(t) \in R^m$ the control input, and $y(t) \in R^p$ the measured output. The system or plant matrix is A, B is the control input matrix, C is the output or measurement matrix, and D is the direct feed matrix. An initial condition vector x(0) and a control input u(t) must be specified to solve the equation.

We sometimes denote the state-space system simply by *(A,B,C,D)*.

To solve this equation in the frequency domain, take the Laplace transform to obtain

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$
.

Now rearrange the state equation to obtain

$$(sI - A)X(s) = x(0) + BU(s)$$
$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$
.

Thus, one also has

$$Y(s) = C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}B + D]U(s).$$

These are the two main equations for solving the state equation. They both have two parts. The first terms depend only on the initial condition x(0) and are the only terms present if the input u(t) is zero. Therefore, they are known as the zero-input (ZI) response. The second terms depend only on the input u(t) and are the only terms present when the initial state is equal to zero. Therefore, they are known as the zero-state (ZS) response. Compare this with the mathematical solution of ordinary differential equations by using homogeneous and particular solutions.

The transfer function is defined by $Y(s) = H(s)U(s)$ when the initial conditions are equal to zero. Therefore, the transfer function is given by

$$H(s) = C(sI - A)^{-1}B + D.$$

The denominator of this is the determinant of (sI-A), denoted

$$\Delta(s) = |sI - A|.$$

The roots of this characteristic polynomial are the system poles. The characteristic equation is

$$\Delta(s) = |sI - A| = 0.$$

The quantity

$$\Phi(s) = (sI - A)^{-1}$$

is known as the resolvent matrix. In terms of the resolvent matrix one may write

$$X = \Phi x(0) + \Phi BU$$

$$Y = C\Phi x(0) + [C\Phi B + D]U = C\Phi x(0) + HU.$$

$$H = C\Phi B + D$$

Note that the output is equal to the transfer function throughput HU plus a part that depends on the initial conditions.

Time Domain Solution

The frequency-domain solution is used to solve the state equation, as shown in a forthcoming example. However, the time-domain solution gives some insight and compares to familiar notions in undergraduate systems courses.

One can see that the resolvent can be written in series form as

$$\Phi(s) = (sI - A)^{-1} = Is^{-1} + As^{-2} + A^2s^{-3} + \cdots,$$

whence a term by term inverse Laplace transform yields

$$[I + At + \frac{A^2t^2}{2!} + \cdots]\, u_{-1}(t)\,,$$

with $u_{-1}(t)$ the unit step. Therefore, one sees that

$$e^{At} = L^{-1}[\Phi(s)]$$

or

$$L[e^{At}] = \Phi(s)\,.$$

This defines the matrix exponential as the inverse Laplace transform of the resolvent matrix. See subsequent example. The matrix exponential is known as the state transition matrix and denoted $\phi(t) = e^{At.}$

Now, one may inverse Laplace transform the state-space solutions for X(s), Y(s) found above to obtain the time-domain solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad + Du(t)\,.$$

(Recall that the product of two Laplace transforms represents convolution in the time domain.) This is reassuringly the same as solutions to differential equations obtained using convolution principles in undergraduate courses.

Recalling the sifting property of the unit impulse (Kronecker delta) $u_0(t)$, one may write the output as

$$y(t) = Ce^{At}x(0) + \int_0^t [Ce^{A(t-\tau)}B + Du_0(t-\tau)]u(\tau)\,d\tau\,.$$

Now recall that the input is convolved with the impulse response to find the output. This identifies the impulse response as

$$h(t) = Ce^{At}B + Du_0(t)\,.$$

Compare this with the definition of the transfer function to see that, as one expects

$$L[h(t)] = H(s)\,.$$

Note that the impulse response is given as the inverse Laplace transform of H(s). To compute the step response r(t), one may simply find

$$r(t) = L^{-1}[H(s)/s]\,.$$

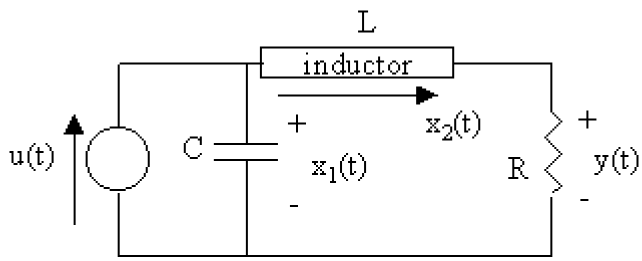
A more general situation occurs when the initial time can take on a general value t_0. Then one has

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad + Du\ .$$

Note that, as in the frequency-domain solution, the solutions have two parts, the ZI part and the ZS part.

Example 1- Electrical Circuit



a. State Equation

Kirchoff's current and voltage laws respectively for this circuit are written down as

$$C\dot{x}_1 = u - x_2$$

$$L\dot{x}_2 = x_1 - Rx_2$$

which yields the state equations directly as

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix}u = \quad Ax + Bu$$

$$y = \begin{bmatrix} 0 & R \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \quad Cx$$

b. Frequency Domain

The characteristic polynomial is

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & 1/C \\ -1/L & s + R/L \end{vmatrix} = s^2 + \frac{R}{L}s + \frac{1}{LC}\ .$$

The resolvent matrix is

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & 1/C \\ -1/L & s + R/L \end{bmatrix}^{-1} = \frac{1}{\Delta(s)}\begin{bmatrix} s + R/L & -1/C \\ 1/L & s \end{bmatrix}$$

The transfer function is

$$H = C\Phi B = \frac{R/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} .$$

c. **Time Constant, Natural Frequency, etc.**

Comparing $\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$ to the standard forms

$$\Delta(s) = s^2 + 2as + w_n{}^2 = s^2 + 2zw_n s + w_n{}^2$$

one sees that

decay term	$a = \frac{R}{2L}$
time constant	$t = \frac{1}{a} = \frac{2L}{R}$
natural frequency	$w_n = \frac{1}{\sqrt{LC}}$
damping ratio	$z = \frac{a}{w_n} = \frac{R}{2}\sqrt{\frac{C}{L}}$
oscillation frequency	$b = \sqrt{w_n{}^2 - a^2} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ $= w_n\sqrt{1-z^2} = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{R^2C}{4L}}$.

d. Time Domain

Selecting values of L= 1 h, R= 3 Ω, C= 0.5 f, one has the characteristic equation
$\Delta(s) = s^2 + 3s + 2 = (s+1)(s+2) = 0$

so the poles are at s= -1, s=-2. Therefore the natural modes are e^{-t}, e^{-2t}. The resolvent matrix becomes

$$\Phi(s) = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}$$

and the transfer function

$$H(s) = \frac{6}{s^2 + 3s + 2} .$$

To find the state transition matrix, one may perform four inverse Laplace transforms, one for each element of Φ(s), to obtain

$$f(t) = e^{At} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} e^{-2t} .$$

Note that the matrix exponential is always expressed in terms of a linear matrix combination of the natural modes. This should technically be multiplied by the unit step $u_{-1}(t)$ since it is causal.

The impulse response is determined by inverse Laplace transform of H(s) to obtain

$$h(t) = (6e^{-t} - 6e^{-2t})u_{-1}(t)$$

It is now desired to find the output y(t) given initial conditions of $x_1(0)=1$ v, $x_2(0)= 2$ A, and an input of $u(t)= 2e^{-3t}u_{-1}(t)$. This is done by computing

$$Y = C\Phi(s)x(0) + HU = \frac{6s+3}{(s+1)(s+2)} + \frac{12}{(s+1)(s+2)(s+3)} .$$

Now an inverse transform yields y(t).

Example 2- Newton's Law System

Newton's third law is *F=ma* or

$$\ddot{y} = \frac{F}{m} \equiv u$$

with *u(t)* the force per unit mass or acceleration input. Selecting the state as $x = [x_1 \quad x_2]^T$ with

$$x_1 = y$$
$$x_2 = \dot{y}$$

one may write the position-velocity state-space equation as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \ddot{y} = u$$

or

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu .$$

The output equation is

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x = Cx$$

which corresponds to position measurements.

a. Poles and Natural Modes.

The characteristic polynomial is

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix} = s^2,$$

so Newton's Law system has two poles at the origin. This makes the two natural modes equal to

$$\begin{aligned} &u_{-1}(t), \quad \textit{unit step} \\ &u_{-2}(t) = tu_{-1}(t), \quad \textit{unit ramp} \end{aligned}.$$

b. Resolvent and STM.

The resolvent matrix is

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}.$$

Inverse Laplace transform this to obtain the state transition matrix

$$f(t) = e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} u_{-1}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{-1}(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u_{-2}(t).$$

Note that $f(t)$ is a sum of the natural modes.

c. Transfer Function.

The transfer function is given by

$$H(s) = C\Phi(s)B = \frac{1}{s^2}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2}.$$

The impulse response is

$$h(t) = tu_{-1}(t),$$

which reflects the fact that an impulsive acceleration on a particle causes its velocity to take on a constant value which makes the position increase linearly.

d. Find State Given ICs and Input.

Find the state if the initial conditions are $x(0) = [s_0 \quad v_0]^T$ and the input acceleration is constant so that $u(t) = g$ *ft/sec*^{2}.

One has

$$X(s) = \Phi(s)x(0) + \Phi(s)BU$$
$$X(s) = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}\begin{bmatrix} s_0 \\ v_0 \end{bmatrix} + \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{g}{s}$$
$$= \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}\begin{bmatrix} s_0 \\ v_0 \end{bmatrix} + g\begin{bmatrix} 1/s^3 \\ 1/s^2 \end{bmatrix}.$$

Now inverse transform to obtain

$$x(t) = \begin{bmatrix} s_0 + v_0 t + \frac{1}{2}gt^2 \\ v_0 + gt \end{bmatrix} u_{-1}(t).$$

This should be a familiar formula from high-school days.

Relative Degree and Zeros of State-Space Systems

The transfer function of a state-space system *(A,B,C,D)* is given by

$$H(s) = C\Phi(s)B + D = C(sI - A)^{-1}B + D$$
$$= \frac{C[adj(sI - A)]B}{|sI - A|} + D = \frac{C[adj(sI - A)]B + D|sI - A|}{|sI - A|} = \frac{N(s)}{\Delta(s)},$$

where *adj(.)* denotes the adjoint of a matrix. Thus, one sees that the poles, which are the roots of the denominator of *H(s),* are given only in terms of *A*. Note that all the information on the feedback loops is contained in *A*. (Think of Mason's Formula.) The zeros generally depend on all four matrices *A,B,C,D.*

The relative degree of *H(s)* is the degree of the denominator minus the relative degree of the numerator. If *A* is an *n x n* matrix, then the degree of $|sI - A|$ is *n*, while the degree of $adj(sI - A)$ is at most *n-1* . Therefore, if *D=0* then the relative degree of *H(s)* must be greater than 1. If *D* is not zero, then the transfer function has relative degree of zero. This means there is a direct feed term.

A transfer function is said to be *proper* if its relative degree is greater than or equal to zero, and *strictly proper* if the relative degree is greater than or equal to one.

Let the system have $x \in R^n, u \in R^m, y \in R^p$. Then the transfer function consists of a *p x m* numerator matrix *N(s)* divided by a scalar denominator $\Delta(s)$. The zeros of the system occur when the matrix *N(s)* loses rank. If the number of inputs equals the number of outputs, *m=p*, then the system is said to be *square*. Then, *N(s)* is a square matrix and the system zeros occur where its determinant vanishes.

If the number of inputs *m* or the number of outputs *p* is greater than one, the system is said to be multi-input/multi-output (MIMO) or *multivariable*. We shall discuss zeros of multivariable systems later if we need to. In the single-input/single-output (SISO) case, things are easier.

Example 3- Zeros of SISO Systems

A system is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu$$
$$y = \begin{bmatrix} a & b \end{bmatrix} x$$
.

The resolvent is equal to

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}.$$

The transfer function is

$$H(s) = C\Phi(s)B = \frac{1}{(s+1)(s+2)} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{bs + a}{(s+1)(s+2)}.$$

Now one sees that the poles are at *s=-1,-2* and the zero is at *s=-a/b*. Thus, the zeros depend on the measurement process, specifically on the gains used in the meters in this example.

If one selects $a/b = 1$ then there is pole/zero cancellation and natural mode e^{-t} is not excited by an impulsive input. If one selects $a/b = 2$ then there is pole/zero cancellation and natural mode e^{-2t} is not excited by an impulsive input.